

2203000205023001
EXAMINATION FEBRUARY -MARCH 2024
BACHELOR OF SCIENCE (FIFTH SEMESTER)
MATHEMATICS-XI (MTH-501-GROUP THEORY) LEVEL 2

[Time: As Per Schedule]

[Max. Marks: 50]

Instructions:

1. Fill up strictly the following details on your answer book

- a. Name of the Examination : **BACHELOR OF SCIENCE (FIFTH SEMESTER)**
 - b. Name of the Subject : **MATHEMATICS-XI (MTH-501-GROUP THEORY) LEVEL 2**
 - c. Subject Code No : **2203000205023001**
2. Sketch neat and labelled diagram wherever necessary.
 3. Figures to the right indicate full marks of the question.
 4. All questions are compulsory.
 5. Follow usual notations.
 6. Use of non-programmable scientific calculator is allowed.

Seat No:

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Student's Signature

Q.1 Answer the following as directed: (Any FIVE)

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- (1) If G is an abelian group, then prove that $(a \cdot b)^2 = a^2 \cdot b^2$; for all a, b in G .
- (2) Define Cyclic Group and Order of an Element in Group.
- (3) If G is a finite group and a in G , then prove that $a^{o(G)} = e$.
- (4) Let H be a subgroup of a cyclic group G ; where $o(G) = 42$ and $o(H) = 21$, then find $i_G(H)$. In this case; what do you say about the subgroup H of a group G ?
- (5) Give an example of a normal subgroup: (i) of an abelian group;
(ii) of a non-abelian group.
- (6) Let H and K be subgroups of orders $o(H) = 30$ and $o(K) = 12$; respectively; with $o(H \cap K) = 6$ of a cyclic group G of order 120. Then find $o(HK)$.
- (7) Express the permutation

$$\theta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 5 & 4 & 3 & 6 & 7 & 8 & 1 & 2 \end{pmatrix}$$

in S_8 as the product of its disjoint cycles.

- (8) Compute $\theta^{-1} \cdot \tau \cdot \theta$; for the permutations
 $\theta = (6\ 3\ 8\ 2) \cdot (5\ 9)$ and $\tau = (1\ 4\ 5\ 3)$ in S_9 .

Q.2 Attempt any TWO:

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- (a) Prove that the set $G = \left\{ \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \mid a, b \in \mathbb{R} \text{ \& } a^2 + b^2 \neq 0 \right\}$ is a group under matrix multiplication.
- (b) Prove that a non-empty subset H of a group G is a subgroup of G if and only if
- (i) $a, b \in H \Rightarrow a \cdot b \in H$,
 - (ii) $a \in H \Rightarrow a^{-1} \in H$.
- (c) Let a be any fixed element in a group G . Then prove that the Normalizer of a ; $N(a) = \{x \in G \mid x \cdot a = a \cdot x\}$ is a subgroup of G .

Q.3 Attempt any TWO:

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- (a) Define Euler's Function ϕ . State Euler's Theorem. If P is a prime number and a is an integer such that $p \nmid a$, then prove that $a^p \equiv a \pmod{p}$.
- (b) If the order of a finite group G is a prime number P , then prove that G is cyclic.
- (c) Let H and K be subgroups of a group G . If HK is a subgroup of G , then prove that $HK = KH$.

Q.4 Attempt any TWO:

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- (a) If N, M are normal subgroups of a group G and $N \cap M = (e)$, then prove that $n \cdot m = m \cdot n$; for every element n in N and for every element m in M .
- (b) Define Homomorphism of Groups. Let $\phi: G \rightarrow \bar{G}$ be a homomorphism of a group G into a group \bar{G} . Then prove that:
- (i) $\phi(e) = \bar{e}$ where e, \bar{e} are the identity elements of groups G and \bar{G} respectively;
 - (ii) $\phi(x^{-1}) = [\phi(x)]^{-1}$; for every x in G .
- (c) (i) If $\phi: G \rightarrow G$ is an automorphism of a group G , then prove that $o(\phi(a)) = o(a)$; for every a in G .
- (ii) Let $G = \langle a \rangle = \{a^0 = e = a^4, a, a^2, a^3\}$ be a cyclic group of order 4 generated by a in G . Then obtain $\mathcal{A}(G)$; the group of all automorphisms of G .

Q.5 Attempt any TWO:

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(a) Prove that the relation $a \equiv_{\theta} b$ on S ; where $a, b \in S$ & $\theta \in A(S)$ is an equivalence relation on S .

(b) (i) Find the Orbit of 3; for the permutation

$$\theta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 9 & 7 & 6 & 5 & 8 & 1 & 4 & 2 & 3 \end{pmatrix} \text{ in } S_9.$$

(ii) Verify whether $\tau^{-1} \cdot \omega$ is even or odd permutation; for the permutations

$$\tau = (3\ 7\ 5) \text{ and } \omega = (4\ 2\ 6\ 1) \text{ in } S_7.$$

(c) Given the permutations; $\sigma = (1\ 3) \cdot (2\ 5)$ and $\tau = (2\ 6) \cdot (3\ 5)$ in S_6 ; find the permutation θ in S_6 satisfying $\theta^{-1} \cdot \sigma \cdot \theta = \tau$.
